EXACT McMAHON SCORE DISTRIBUTION

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September 2016

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References

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1 MOTIVATION

Monte-carlo simulation methods are used to extract characteristics of Go tournaments. The accuracy of the simulation depends on the number of samples used to calculate values of interest. An exact calculation of the score distribution is therfore useful for estimating the sample size required.

Tournament pairing in Go is quite complex. This article aims to provide an analytic solution for the score distribution in small Swiss and McMahon tournaments. With the simplest pairing rules, the problem of finding the score distribution analytically is tractable for the case of two-round tournaments with either four or six players. The principles developed apply to more players or rounds, but the combinations of pairings and results become far too numerous for manual manipulation.

The plan is to systematically generate every possible tournament for a fixed set of players with specified grades. Using the known win probability [1] for player pairs, we can then generate the probability for any particular tournament.

Pairing in Swiss and McMahon tournaments is governed by strict rules and we will examine every possible pairing for rounds 1 and 2, the latter of course depending on the game results for round 1. We first however discuss the method for generating game results and their probabilities, since here at least the procedure is quite general and applies to any number of rounds or players.

2 SCORE VALUES AND PROBABILITIES

We are given a pairing for an even number $n = 2m$ of players. For each pair, the outcome of the game is 1 if the first of the pair wins, and 0 if the second of the pair wins. Since each pair has an outcome of 0 or 1, we can regard the list of game results as forming a binary sequence of length m. The number of possible result sequences for the given pairing is then 2^m. We generate all possible results as the patterns in the binary expansion of an integer u in the range $[0, 2^m - 1]$.

The score probability for a pair(i, j) with given grades $\mathsf{g}_{ij} = (g_i, g_j)$ is from Appendix A.2, Equation (12):

$$p_{\text{score}}(i, j, s) = p_{\text{grade-score}}(\mathsf{s} \mid \mathsf{g}_{ij})$$

A pairing of n players can be expressed in the form:

$$\mathbf{d} = i_0 j_0, i_1 j_1, \ldots\ i_{m-1} j_{m-1}$$

The scores on the pairs are mutually independent. Suppose s_b is the b^{th} bit in the binary expansion of the integer s. Then the scoring probability for the draw (Appendix A.4, Equation (19)) can be expressed in the form:

$$p_{\text{score}}(\mathbf{s} \mid \mathbf{d}) = \prod_{b=0}^{b=m-1} p_{\text{grade-score}}(\mathsf{s}_b \mid \mathsf{g}_b) \tag{1}$$

3 FOUR PLAYER SWISS

The simplest Swiss pairing rules are:

- No byes.
- No repeat pairing.
- Each player starts on score 0.
- A player's score increases by 1 for each win.
- Pair players on minimum score difference.

The pairing for any round can be presented as a permutation of the sequence 1234. The pairings 1234 and 3421 are the same, and we avoid these repeats by the rules:

- The first of a pair has the lowest player number of the pair.
- Pairs are ordered by the first player number.

3.1 At the end of round one

The first round is paired at random, and we can choose the first pair in 3 ways; the second pair is then fixed. With the convention described above, the three possible first round pairings are $12\,34, 13\,24, 14\,23$. This means that the first round pairings have one degree of freedom i.e. a single variable d can fully specify the pairing via the table:

d	draw(**d**)
1	12 34
2	13 24
3	14 23

Table 1: Round 1 pairing table

So the set of draws $\mathbb{D}_{\tau_0}$ generated from the starting tournament τ_0 has three elements. Since all the draws are equally likely, the probability of any draw $\mathbf{d}$ is a constant:

$$p_{\text{draw}}(\mathbf{d}_1 \mid \tau_0) = P_{\mathbb{D}_{\tau_0}}(\mathbf{d}) = \frac{1}{3} \tag{2}$$

Substituting Equations (1) and (2), in Equation (27) of Appendix A.6 we find:

$$\begin{aligned} \tau_1 &= ((\tau_0, \mathbf{d}), \mathbf{s}) \\ p^1_{\text{tour}}(\tau_1) &= p_{\text{score}}(\mathbf{s} \mid \mathbf{d}) P_{d_1} \\ P_{d_1} &= \frac{1}{3} \end{aligned} \tag{3}$$

The value for $p_{\text{score}}(\mathbf{s} \mid \mathbf{d})$ is given in Equation (1).

3.2 Pairing and results for round two

For the second round pairing, we sort the players by their first round scores. This gives two players on 0 and two on 1. The players on the same score cannot have played each other in the first round. So there is just one pairing $\mathbf{d}_2$ for the second round satisfying the Swiss rules and there are no degrees of freedom. The set $\mathbb{D}_{\tau_1}$ has just one element, hence

$$p_{\text{draw}}(\mathbf{d}_2 \mid \tau_1) = P_{\mathbb{D}_{\tau_1}}(\mathbf{d}_2) = 1 \tag{4}$$

Although the draw for the second round depends on the draw and results for the first round, we assume that the scoring probabilities for the second round depend only on the second round draw. Again applying the recurrence Equation (27), we obtain from (4) the tournament probability for the 2-round, 4-player Swiss as:

$$\begin{aligned} \tau_2 &= ((\tau_1, \mathbf{d}_2), \mathbf{s}_2) \\ p^2_{\text{tour}}(\tau_2) &= p_{\text{score}}(\mathbf{s}_2 \mid \mathbf{d}_2) P_{d_2}\, p^1_{\text{tour}}(\tau_1) \\ P_{d_2} &= 1 \end{aligned} \tag{5}$$

The number of results for each round is $2^2 = 4$, and so the total number of different Swiss tournaments with 2 rounds and 4 players is:

$$T^4_2 = 3 \times 4 \times 4 = 48$$

Once the grades for all players have been specified, we can generate all the above tournaments along with their joint probabilities using Equations (3) and (5).

4 SIX PLAYER SWISS

The scoring probabilities for either round are evaluated exactly as in Equation (1), and the same recurrence relation in Equation (27) is used to develop the probability for any possible tournament. The main difference in the 6-player Swiss from the 4-player Swiss lies in the evaluation of the draw probabilities for the two rounds.

4.1 Draw for round one

The pairing for round 1 is random. There are 5 ways to choose the first pair, leaving 4 players to be chosen for the second pair. The draw thus has two degrees of freedom specified by the two integer variables α in the range $A = [1, 5]$ for the first pair and β in the range $B = [1, 3]$ for the second pair (as in section 3.1). We can therefore construct a draw generating function dgen_1 which produces a unique draw from α, β. This means we can determine the set $\mathbb{D}_{\tau_0}$:

$$\mathbb{D}_{\tau_0} = \{\, \mathbf{d} = \mathsf{dgen}_1(\alpha, \beta) : \alpha \in A \wedge \beta \in B \,\} \tag{6}$$

All the draws generated by dgen_1 are equally likely, so the draw probability is the constant:

$$p_{\text{draw}}(\mathbf{d}_1 \mid \tau_0) = P_{\mathbb{D}_{\tau_0}}(\mathbf{d}) = \frac{1}{5} \times \frac{1}{3} \tag{7}$$

4.2 At the end of round two

On completion of round 1, we have 3 players on score $\mathsf{0}$ and 3 players on score $\mathsf{1}$. The best pairing we can achieve is to pair one of the losers with one of the winners. We can choose a loser in 3 ways, but only two of the winners is available for a game. The pairing in the second round has two degrees of freedom represented by integer variables μ in the range $U = [1, 3]$ for the 3 losers and ν in the range $V = [1, 2]$ for the 2 available winners. Again we can define a draw generating function dgen_2 to produce the set of draws for the second round:

$$\mathbb{D}_{\tau_1} = \{\, \mathbf{d} = \mathsf{dgen}_2(\mu, \nu) : \mu \in U \wedge \nu \in V \,\} \tag{8}$$

All the draws arising from the free choice of μ and ν are equally likely. Applying the recurrence Equation (27) to round 1 then round 2, we find the probability of tour τ_2 at the end of round 2 in the form:

$$\begin{aligned} \tau_2 &= ((\tau_1, \mathbf{d}_2), \mathbf{s}_2) \\ p^2_{\text{tour}}(\tau_2) &= p_{\text{score}}(\mathbf{s}_2 \mid \mathbf{d}_2) P_{d_2}\, p_{\text{score}}(\mathbf{s}_1 \mid \mathbf{d}_1)\, P_{d_1} \\ P_{d_2} &= \frac{1}{3} \times \frac{1}{2}, \quad P_{d_1} = \frac{1}{5} \times \frac{1}{3} \end{aligned} \tag{9}$$

The total number of results for each round is $2^3 = 8$, so the total number of distinct Swiss tournaments of 2 rounds with 6 given players is:

$$T^2_6 = 5 \times 3 \times 8 \times 3 \times 2 \times 8 = 5760$$

5 APPLICATION TO McMAHON

5.1 Pairing rules

The simple rules used for the above Swiss tournaments are extended to provide simple McMahon tournaments:

- No byes.
- No repeat pairing.
- Each player is given an initial McMahon score.
- The player's McMahon score increases by 1 for each win.
- Pair players on minimum McMahon score difference.

In real-life McMahon tournaments, the initial McMahon score (denoted mmi) increases by 1 for each grade step, apart from players in the top group who all get the same mmi. In order to apply the results developed above for Swiss pairing, we modify the assignment of mmi:

1. The entry consists of equally sized groups of players, $G_0, G_1, \ldots G_M$.
2. Each group has players of possibly different grades.
3. The group G_0 is homogenous, and its mmi is set according to the group grade.
4. For any other group m, the group mmi is set by $\mathsf{mmi}_m = \mathsf{mmi}_0 + 3m$.

Since there are only 2 rounds, the highest McMahon score in any group is $\mathsf{mmi}_m + 2$. The above mmi rule ensures that players from different groups should not be paired. Each group then behaves as a mini Swiss tournament within the McMahon tournament.

The pairing method for the above McMahon tournament is based on a weighted matching algorithm presented in [2]. The only information the algorithm uses for a pairing is the current McMahon score and the list of available opponents. The algorithm does not 'know' it is running a collection of mini Swiss tournaments.

5.2 Flavour

Within any one of the above groups, players of different grades meet on even terms. The resulting player performance will depend on the nature of the distribution of grades in the group. Call this the *flavour* of the group.

partition	grade flavour	mmi
1+1+1+1	3d 4d 5d 6d	2
1+1+2	1k 1d 2d	-1
2+2	**4k 3k**	-4
1+3	7k 6k	-7
4	10k	-10

Table 2: Grade flavour for 4 players

The group can be partioned into subsets of players with the same grade. The number of possible partitions depends on the group size: for 4 players it is 5 and for 6 players it is 11. In setting up the enry for the McMahon tournament, the grade of each flavour in the group increases by 1. The grade distribution of each group in the tournament is shown in the grade flavour tables 2 and 3. For each partition we list the grades in the same order as the partition size. The table also shows the mmi for the group.

In the 4 player groups shown in Table 2 the grades are never more than 1 apart for the bottom 3 flavours. The top group has a maximum grade differential of 3 grades. There is more variety in the 6 player groups shown in Table 3, and the maximum grade difference rises to 5 grades for the top group.

partition	grade flavour	mms_i
1+1+1+1+1+1	1d 2d 3d 4d 5d 6d	0
1+1+1+1+2	3k 2k 1k 1d 2d	-3
1+1+2+2	6k 5k 4k 3k	-6
1+1+1+3	9k 8k 7k 6k	-9
2+2+2	12k 11k 10k	-12
1+2+3	15k 14k 13k	-15
1+1+4	18k 17k 16k	-18
3+3	21k 20k	-21
2+**4**	24k **23k**	-24
1+5	27k 26k	-27
6	*30k*	-30

Table 3: Grade flavour for 6 players

5.3 Monte Carlo simulation

The Monte-carlo simulation follows this process:

Algorithm mm1. `mcmahon-sim`

msim-1 Generate the entry covering all flavours.

msim-2 Simulate both rounds.

msim-3 Update statistics.

msim-4 Continue at msim-2.

In step msim-2 we pair round 1, simulate results according to $p_{\text{grade-score}}(\mathsf{s}_{ij} \mid \mathsf{g}_{ij})$, pair round 2, and simulate results for round 2. This gives a complete tournament for feeding into step msim-3. The simulation was repeated for a wide range in the number of samples, and all the data collected is available in [5].

The win probability for a grade-pair $\mathsf{g} = (r, s)$ used for the Monte Carlo simulation (and the analytic calculation) is given [1] by:

$$p_{\text{grade-score}}(\mathtt{1} \mid \mathsf{g}) = \tfrac{1}{2}[1 - \operatorname{erf}(\Lambda(g_i, g_j))]$$

$$\Lambda(r, s) = \sum_{n=0}^{3} h_n(s - r) e^{n\,K\,\min(r,s)}$$

$$h_n(x) = u_n x + v_n x^3, \quad n = 0 \cdots 3$$

	$\mathbf{u_0}$	$\mathbf{v_0}$	$\mathbf{u_1}$	$\mathbf{u_3}$	**K**
solution	0.0351224	0.00445376	0.156777	0.0164481	0.18818

Table 4: Coefficients for win probability

The non-zero constants u_n and v_n are given in Table 4

The weight ω_{ij} for the pairing between players i, j with grades $\mathsf{g}_i, \mathsf{g}_j$, in the Monte Carlo simulation is set [2] according to:

$$\omega_{ij} = W_0 + W_{\mathsf{mms}} \mathrm{sech}(\lambda_{\mathsf{mms}}(\mathsf{g}_i - \mathsf{g}_j))$$
$$W_0 = 2^{39}, \quad W_{\mathsf{mms}} = 2.0 \times 2^{32}, \quad \lambda_{\mathsf{mms}} = 0.1$$

6 MONTE-CARLO ACCURACY

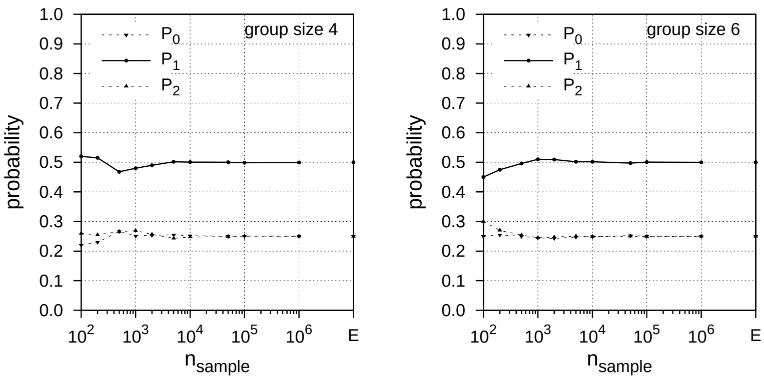

Figure 1: Scoring probability for homogenous groups

The plot in Figure 1 shows how the score probabilities[1] approach the theoretical value for homogenous groups as the number of samples is increased from 100 to 1 million. Analytic values are placed at the point E. The values for score probability are accurate to better than 0.3% for $n_{\texttt{sample}} > 10^4$.

The lowest graded groups in the simulation are homogenous.

P_0	P_1	P_2	mean
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

Table 5: Homogenous group statistics

[1] P_r refers to the number r of wins, rather than McMahon score

Each player in these groups plays two others of the same grade, and each game has a probability of win of $\frac{1}{2}$ for either player. There is one way for a player to score 0 or 2, and 2 ways to score 1. So we can immediately write down the score probability distribution and its mean for any player, as given in Table 5.

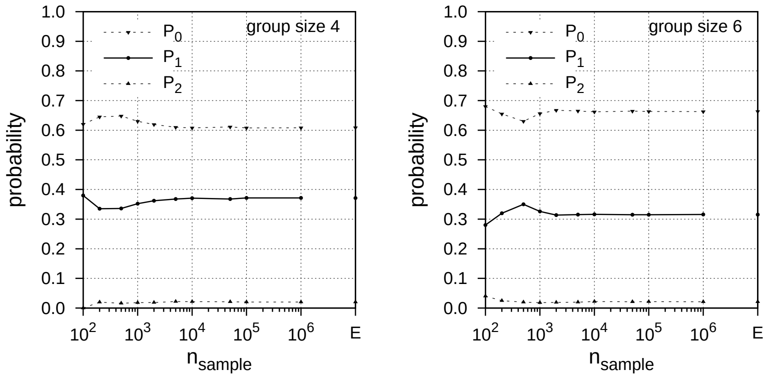

Figure 2: Scoring probability for top groups

The top group is extreme in that no two players have the same grade. The simulation results for the top groups in Figure 2 show however that the Monte-Carlo accuracy is not sensitive to the grade distribution in these very different kinds of groups.

The results of the analytic calculation for the scoring probability of every player is shown in Appendix B. Apart from the homogenous groups, there are special groups having the property[2] that the probability of winning just one game is *exactly* $\frac{1}{2}$. The bold entries in Tables 2 and 3 identify these groups.

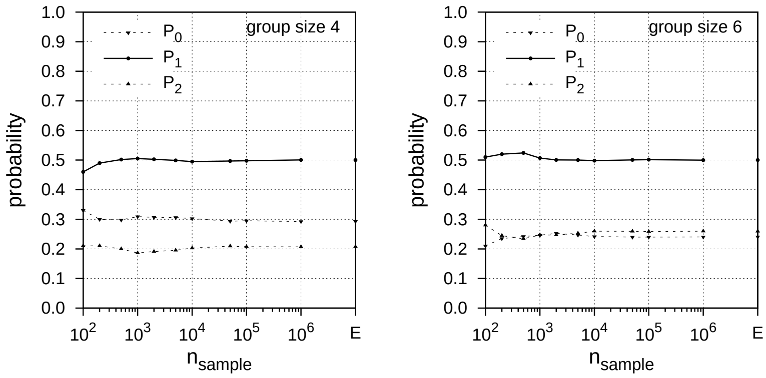

Figure 3: Scoring statistics for special groups

[2]This is a feature of the flavour, independent of actual win probabilities

For the four-player group with flavour 2+2, the grades are balanced. For the six-player group with flavour 2+4, it is the 4 player sub-group that is special. Note that the 3+3 group does not share this property. The simulation results for these special cases are shown in Figure 3. Again the simulation acuracy is not sensitive to the special nature of these groups.

Since any player's mean score is obtained as the sum $p_{\text{score}}(1) + 2\,p_{\text{score}}(2)$, it follows that the behaviour of the mean score is similar to the behaviour of the score probabilities for large $n_{\texttt{sample}}$.

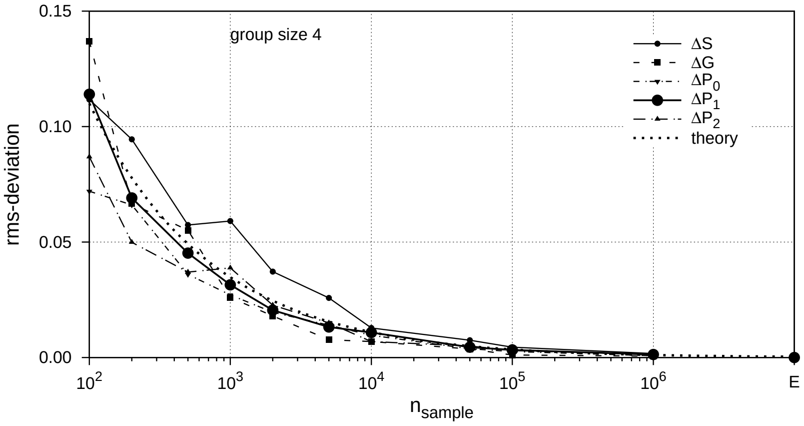

Figure 4: Deviation for 4 player groups

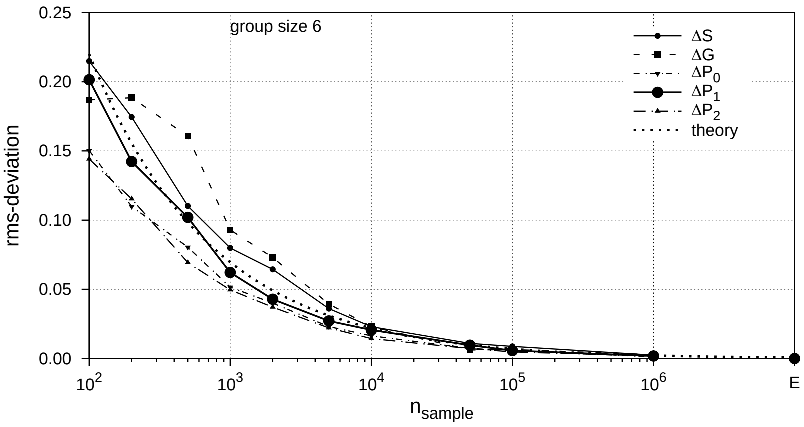

Figure 5: Deviation for 6 player groups

So far we have highlighted the behaviour of the distributions for particular players in groups of interest. We can get an overview of the simulation accuracy for *all* players by considering the root-mean square deviation of the simulation result from the analytic result.

This deviation is presented in Figures 4 and 5. The graph marked ΔS shows the deviation in the mean score. The graph ΔG is the deviation in the mean grade of the opponents for each player. This quantity is of interest for the bar setting algorithm discussed in [4].

The graph marked *theory* in these plots is an estimated fit of the $A/\sqrt{n}$ accuracy law [3] which holds for Monte Carlo integration. It is interesting to note that the *theory* graph fits well with the score probability $p_{\text{score}}(1)$, and is a plausible fit for the mean score. Rules such as this might lead to efficient ways to estimate the number of samples required to achieve a specified accuracy for the simulation of any McMahon tournament.

7 CONCLUSION

An exact joint distribution for player score is developed for 2-round Swiss tournaments of 4 or 6 players. The distribution is valid for any predefined win probability between two players of any rating.

The assumptions made for the distribution are:

- In any round individual player scores are independent.
- The score distribution is independent of past history.
- Within a round all possible draws are equally likely.

The tournament distribution presented as a recurrence relation makes it easier to model alternative pairing schemes which may break the above assumptions.

Monte Carlo simulation of a McMahon tournament constructed as a collection of mini Swiss tournaments, confirms that measured probabilities converge to the analytic probabilities.

The pairing algorithm used in the Monte Carlo simulation shuffles the player list before feeding to the maximum weighted matching algorithm. The convergence results confirm that this shuffling is sufficient to eliminate any bias that may be induced by the matching algorithm.

A TOURNAMENT PROBABILITY

A.1 Notation and definitions

A *probability mass function* (pmf) for any finite set X is a function P_X mapping the power set $\mathcal{P}(X)$ into the real unit interval $\mathbb{U} = [0, 1]$ with the properties:

$$\begin{aligned} P_X(A) &= \sum_{a \in A} P_X(\{a\}), \texttt{ for any } A \subseteq X \\ P_X(\varnothing) &= 0 \\ P_X(X) &= 1 \end{aligned}$$

The set X is the *sample-space* for P_X, and where there is no ambiguity, we write the probability of a singleton as $P_X(\{x\}) = P(x)$.

If A is a subset of X then A *inherits* a pmf P_A defined by:

$$P_A(a) = P_X(a)/P_X(A) \quad \forall\, a \in A$$

Suppose f is a function mapping X into some set Z. Then $\mathrm{f}[X]$ denotes the *image set* of X defined as $f[X] = \{\mathrm{f}(x) : x \in X\}$. Ley $Y = \mathrm{f}[X]$. The function f *induces* a pmf P_Y for Y given by:

$$\begin{aligned} X_y &= \{x : y \in Y \wedge \mathrm{f}(x) = y\} \\ P_Y(y) &= P_X(X_y)\, \forall\, y \in Y \end{aligned} \tag{10}$$

Suppose T is a multi-dimensional cartesian product of a sample space X i.e. $T = X^m$. Then any point in T is $\mathbf{x} = (x_1 \cdots x_m)$. For each $\nu = 1 \cdots m$, define a *co-ordinate projection* $\mathsf{L}_\nu : T \to X$ and an associated family $\mathcal{T}^\nu$ of *co-ordinate slices* by:

$$\begin{aligned} \mathsf{L}_\nu(\mathbf{x}) &= x_\nu && (11) \\ T^\nu(q) &= \{\mathbf{x} \in T : \mathsf{L}_\nu(\mathbf{x}) = q\} \quad q \in X \\ \mathcal{T}^\nu &= \{T^\nu(q) : q \in X\} \end{aligned}$$

A.2 EGD data

The tables provided by the European Go Database [6] give empirical values for the probability of win in a game between players of given grades. Translating the results to the formalism above, let G be a finite subset of $\mathbb{R} \times \mathbb{R}$ representing the two grades (or ratings) of the players in a game. Player grades are usually expressed in integral terms, but the theory developed here applies equally to ratings expressed as real numbers scaled to units of grade. Each $\mathsf{g} = (g_1, g_2) \in G$ is a *rated-pair*.

The possible results of a game between two players is represented by the sample space $\mathbb{S} = \{\mathsf{0}, \mathsf{1}\}$ where $\mathsf{0} = (0, 1)$ and $\mathsf{1} = (1, 0)$. Each $\mathsf{s} \in \mathbb{S}$ is a *game-score* and we call the pair $(\mathtt{g}, \mathtt{s})$ a *rated-game*. The components (s_1, s_2) of $\mathtt{s}$ provide the result of the game for each component grade in $\mathtt{g}$: i.e. g_i achieves result s_i for $i = 0, 1$.

The tables in E.G.D give values for the following quantities:

symbol	description
$\mathcal{G}$	a set of rated-pairs $\mathtt{g} = (g_1, g_2)$ with $g_1 < g_2 \leq g_1 + 4$.
$n_{\mathtt{game}}(\mathtt{g})$	the number of rated-games with grade-pair $\mathtt{g}$.
$n_{\mathtt{win}}(\mathtt{g})$	the number of rated-games with grade-pair $\mathtt{g}$ and game-score $\mathtt{1}$

Table 6: E.G.D data

Let N_G be the total number of rated-games collected. The data provides values for the folowing functions:

$$p_{\mathtt{grade}}(\mathtt{g}) = n_{\mathtt{game}}(\mathtt{g})/N_G$$
$$p_{\mathtt{win}}(\mathtt{g}) = n_{\mathtt{win}}(\mathtt{g})/N_G$$
$$p_{\mathtt{win/grade}}(\mathtt{g}) = n_{\mathtt{win}}(\mathtt{g})/n_{\mathtt{game}}(\mathtt{g})$$

As it stands, the above tables cover the grade range from 20 kyu to 7 dan. The range has been extended [1] to any score and real grade via fitted functions:

$$p_{\text{grade-score}}(\mathtt{g}, \mathtt{1}) = p_{\mathtt{win}}(\mathtt{g})$$
$$p_{\text{grade-score}}(\mathtt{g}, \mathtt{0}) = p_{\mathtt{grade}}(\mathtt{g}) - p_{\mathtt{win}}(\mathtt{g})$$

Let $\mathbb{W} = \{\mathtt{g}\} \times \mathbb{S}$ be the sample space for rated-games. The conditional probability of the score for a rated-game $(\mathtt{g}, \mathtt{s})$, given a rated-pair $\mathtt{g}$ is $P_{\mathbb{W}}(\mathtt{s} \mid \mathtt{g})$ written as $p_{\text{grade-score}}(\mathtt{s} \mid \mathtt{g})$ and defined by:

$$p_{\text{grade-score}}(\mathtt{1} \mid \mathtt{g}) = p_{\mathtt{win/grade}}(\mathtt{g}) \tag{12}$$
$$p_{\text{grade-score}}(\mathtt{0} \mid \mathtt{g}) = 1 - p_{\mathtt{win/grade}}(\mathtt{g})$$

In applying (12), we assume that the values for $p_{\text{grade-score}}(\mathtt{s} \mid \mathtt{g})$ do not depend on the particular collection of rated-pairs G used in E.G.D.

A.3 Tournament Structure

A *tournament* τ is a sequence of *rounds*, where the possible *draw* for a round depends on the draw and score for each previous round. The set $\mathbb{D}_\tau$ denotes the set of all possible pairings that can be produced from τ. A *draw* is an ordered sequence of *game-pairs*, with ordering defined in Section 3. A game-pair consists of two players chosen from the tournament entry E (with $n = 2m$ players). Each *scored-game* consists of a game-pair and game-score. Each round $\mathbf{h}$ is a sequence of *scored-games*. Player grades are obtained from a grade function G, which determines a grade-pair from a game-pair Formally:

equation	meaning
$\mathsf{d} = (i,j), \quad i<j, \quad i,j \in E$	game-pair
$\mathbb{S} = \{\mathsf{0},\mathsf{1}\} \quad \mathsf{0} = (0,1),\ \mathsf{1} = (1,0)$	scores
$\mathsf{h} = (\mathsf{d},\mathsf{s}), \quad \mathsf{s} \in \mathbb{S}$	scored-game
$\mathsf{g} = G(\mathsf{d})$	grade-pair
$\mathbf{h} = (\mathsf{h}_1 \cdots \mathsf{h}_m)$	round

Table 7: Tournament elements

A draw is a sequence $\mathbf{d} = (\mathsf{d}_1 \cdots \mathsf{d}_m)$. A round $\mathbf{h}$ can be composed from a draw $\mathbf{d}$ and a *score-sequence* $\mathbf{s} = (\mathsf{s}_1 \cdots \mathsf{s}_m)$ via the composition operator $\odot$ defined by:

$$\begin{aligned} \mathsf{d}_\nu &= \mathsf{L}_\nu(\mathbf{d}), \quad \mathsf{s}_\nu = \mathsf{L}_\nu(\mathbf{s}) \\ \mathsf{h}_\nu &= (\mathsf{d}_\nu, \mathsf{s}_\nu), \quad \nu = 1 \cdots m \\ \mathbf{d} \odot \mathbf{s} &= (\mathsf{h}_1 \cdots \mathsf{h}_m) \end{aligned} \tag{13}$$

A.4 Scoring Probability

We consider here a tournament τ completed to the k^{th} round. Let $\mathbf{d} \in \mathbb{D}_\tau$ be a draw for the next round. Associated with each component d_ν of $\mathbf{d}$ is a sample space $\mathbb{S}_\nu$ for the scored-games introduced in Table 7. With a round $\mathbf{h}$ defined as a sequence of scored-games, we find that the set of all rounds $H_\mathbf{d}$ can be written as a cartesian product of the individual $\mathbb{S}_\nu$:

$$\begin{aligned} \mathsf{d}_\nu &= \mathsf{L}_\nu(\mathbf{d}), \quad \nu = 1 \cdots m \\ \mathbb{S}_\nu &= \{\mathsf{d}_\nu\} \times \mathbb{S} \\ H_\mathbf{d} &= \mathbb{S}_1 \times \mathbb{S}_2 \cdots \times \mathbb{S}_m \end{aligned}$$

In this section $\mathbf{d}$ is fixed, so from now on let R denote the set of rounds $H_\mathbf{d}$. Let $\mathbf{s}$ be a score sequence, and define co-ordinate slices:

$$R^\nu_{\mathsf{s}_\nu} = \{\, \mathbf{h} \in R : \mathsf{L}_\nu(\mathbf{h}) = (\mathsf{d}_\nu, \mathsf{s}_\nu)\,\} \quad \nu = 1 \cdots m \tag{14}$$

Then the intersection of these slices is the composition of the draw $\mathbf{d}$ and $\mathbf{s}$:

$$R_\mathbf{s} = \bigcap_{\nu=1}^{\nu=m} \{R^\nu_{\mathsf{s}_\nu}\} = \{\mathbf{d} \odot \mathbf{s}\} \tag{15}$$

We assume that the scores s_ν for the player-pairs are independent, so we require that the family $\{\, R^\nu_{\mathsf{s}_\nu} : \nu = 1 \cdots m \,\}$ is mutually independent. With P_R a pmf for R, the scoring probability for $\mathbf{h} = \mathbf{d} \odot \mathbf{s}$ can be written as a product:

$$P_R(\mathbf{h}) = P_R(R_\mathbf{s}) = \prod_{\nu=1}^{\nu=m} P_R(R^\nu_{\mathsf{s}_\nu}) \tag{16}$$

The co-ordinate projection $\mathsf{L}_\nu : R \to \mathbb{S}_\nu$ induces a pmf on each component space $\mathbb{S}_\nu$ given by:

$$\hat{P}_{\mathbb{S}_\nu}(\mathsf{d}_\nu, \mathsf{s}) = P_R(R^\nu_{\mathsf{s}}), \quad \nu = 1 \cdots m \tag{17}$$

The score probability for each rated-game is expressed in terms of the space $\mathbb{W}_\nu = \{\mathsf{g}_\nu\} \times \mathbb{S}$ introduced in Section A.2. The grade-pair g_ν for each player-pair is determined from $\mathsf{g}_\nu = G(\mathsf{d}_\nu)$. We can map $\mathbb{S}_\nu$ onto $\mathbb{W}_\nu$ via the fuction $\mathsf{f}(\mathsf{d}_\nu, \mathsf{s}) = (G(\mathsf{d}_\nu), \mathsf{s})$. This induces a pmf $\hat{P}_{\mathbb{W}_\nu}$ on $\mathbb{W}_\nu$ given by:

$$\hat{P}_{\mathbb{W}_\nu}(\mathsf{g}_\nu, \mathsf{s}) = \hat{P}_{\mathbb{S}_\nu}(\mathsf{d}_\nu, \mathsf{s}) \tag{18}$$

We require that the induced pmf $\hat{P}_{\mathbb{W}_\nu}$ is the same as the known pmf $P_{\mathbb{W}_\nu}$. From Equations (16), (17), (18), and (12) we get:

$$\begin{aligned} P_R(R^\nu_{\mathsf{s}_\nu}) &= \hat{P}_{\mathbb{S}_\nu}(\mathsf{d}_\nu, \mathsf{s}_\nu) = \hat{P}_{\mathbb{W}_\nu}(\mathsf{g}_\nu, \mathsf{s}_\nu) \\ &= P_{\mathbb{W}}(\mathsf{s}_\nu \mid \mathsf{g}_\nu) \equiv p_{\text{grade-score}}(\, \mathsf{s}_\nu \mid \mathsf{g}_\nu) \end{aligned}$$

Finally we obtain the scoring probability for a round as:

$$\begin{aligned} P_R(\mathbf{h}) &= \prod_{\nu=1}^{\nu=m} p_{\text{grade-score}}(\, \mathsf{s}_\nu \mid \mathsf{g}_\nu) \\ \mathsf{L}_\nu(\mathbf{h}) &= (\, \mathsf{d}_\nu, \mathsf{s}_\nu), \quad \nu = 1 \cdots m \\ \mathsf{g}_\nu &= G(\mathsf{d}_\nu) \end{aligned} \tag{19}$$

A.5 Pairing Probability

T is the set of tournaments completed to round k, with given pmf P_T. $\mathbb{D}_\tau$ is the set of possible draws for the next round in tournament $\tau \in T$, and $\mathbb{D}_T = \bigcup\{\mathbb{D}_\tau : \tau \in T\}$ is the set of all possible next-round draws for all tours in T. We assume that the pmf $P_{\mathbb{D}_\tau}$ is given.

For each $\tau \in T$, let $D_\tau = \{\tau\} \times \mathbb{D}_\tau$. Then the sample space W for all next-round draws in all tournaments in T is given by:

$$W = \bigcup\{D_t : t \in T\} \tag{20}$$

This means that the co-ordinate slice W_τ through τ is just D_τ. Let $W_{\mathbf{d}}$ be a co-ordinate slice of W through the point $\mathbf{d} \in \mathbb{D}_T$ defined by:

$$W_{\mathbf{d}} = \{\, (t, \mathbf{d}) : t \in T \quad \wedge \quad \mathbf{d} \in D_t \,\}$$

It follows from these definitions that the pmf P_W is given by:

$$P_W(\tau, \mathbf{d}) = P_W(W_\tau \cap W_{\mathbf{d}}) = P_W(W_{\mathbf{d}} \mid W_\tau) P_W(W_\tau) \tag{21}$$

The first term $P_W(W_{\mathbf{d}} \mid W_\tau)$ of (21) induces a pmf for $\mathbb{D}_\tau$.

Firstly, the slice W_τ is a subset of W so it inherits a pmf from W:

$$P_{W_\tau}(\tau, \mathbf{d}) \underset{\mathsf{def}}{=} P_W(\tau, \mathbf{d})/P_W(W_\tau) = P_W(W_\mathbf{d} \mid W_\tau)$$

Secondly, define a mapping $\Omega_\tau : W_\tau \to \mathbb{D}_\tau$ by $\Omega_\tau(\tau, \mathbf{d}) = \mathbf{d}$. This mapping[3] induces a pmf on $\mathbb{D}_\tau$:

$$\hat{P}_{\mathbb{D}_\tau}(\mathbf{d}) = P_{W_\tau}(\tau, \mathbf{d}) \quad \forall\, \mathbf{d} \in \mathbb{D}_\tau$$

We require that P_W satisfies the condition that the induced $\hat{P}_{\mathbb{D}_\tau}$ is the same as the given $P_{\mathbb{D}_\tau}$:

$$\begin{aligned} P_W(W_\mathbf{d} \mid W_\tau) = P_{W_\tau}(\tau, \mathbf{d}) &= \hat{P}_{\mathbb{D}_\tau}(\mathbf{d}) \\ &= P_{\mathbb{D}_\tau}(\mathbf{d}) \quad \forall\, \mathbf{d} \in \mathbb{D}_\tau \end{aligned} \tag{22}$$

The second term $P_W(W_\tau)$ of (21) induces a pmf for T.

Consider the map $\Gamma : W \to T$ defined by $\Gamma(\tau, \mathbf{d}) = \tau\ \forall\, (\tau, \mathbf{d}) \in W$. Since $\Gamma[W_\tau] = \{\tau\}$, we see that Γ induces a pmf on T defined by:

$$\hat{P}_T(\tau) = P_W(W_\tau) \quad \forall\, \tau \in T$$

We require that the induced pmf on T is the same as the given pmf P_T i.e.

$$P_W(W_\tau) = P_T(\tau) \quad \forall\, \tau \in T \tag{23}$$

A.6 Next-Tournament Probability

We develop a recurrence relation for the probability of a tournament at the next round given the current tournament, next-round draw, and next-round score. As discussed in the previous section, W is the sample space for the next draw $\mathbf{d}$ in the current tournament τ. So, for a given score $\mathbf{s} \in S$, the next tournament is the pair $(w, \mathbf{s})$, where $w = (\tau, \mathbf{d}) \in W$.

It follows that a sample space for all the next tournaments obtainable from any current tournament is the set $Z = W \times S$. We will express the pmf P_Z in terms of the pmf's P_W and $P_{H_\mathbf{d}}$ already established for the probabilities of draw and score.

Suppose $z = (w, \mathbf{s}) \in Z$ is the next tournament. It then follows[3] that there exists a current tournament $\tau \in T$, a draw $\mathbf{d} \in \mathbb{D}_\tau$ for τ, and a round $\mathbf{h} \in H_\mathbf{d}$ satisfying:

$$w = (\tau, \mathbf{d}), \quad \mathbf{h} = \mathbf{d} \odot \mathbf{s} \tag{24}$$

We introduce two co-ordinate slices through the point $(w, \mathbf{s}) \in Z$:

$$Z_w = \{w\} \times S, \quad Z_\mathbf{s} = W \times \{\mathbf{s}\}$$

The pmf P_Z can then be decomposed:

$$P_Z(w, \mathbf{s}) = P_Z(Z_w \cap Z_\mathbf{s}) = P_Z(Z_\mathbf{s} \mid Z_w) P_Z(Z_w) \tag{25}$$

[3] $w = (\tau, \mathbf{d}) \in W \Rightarrow \tau \in T$ and $\mathbf{d} \in \mathbb{D}_\tau$

The probability for the next draw is obtained from Z_w.

The projection $\Omega : Z \to W$ defined by $\Omega(w, \mathbf{s}) = w$ induces a pmf on W given by $\hat{P}_W(w) = P_Z(Z_w)$. We require that $\hat{P}_W$ is the same as the pmf P_W discussed in Section A.5.

The next round scoring probability is found from $P_Z(Z_{\mathbf{s}} \mid Z_w)$ in (25).

The set Z_w contains all the scores for a given draw $\mathbf{d}$ in the tournament τ. The conditional defines an inherited pmf $P_{Z_w}(\mathbf{s}) = P_Z(Z_{\mathbf{s}} \mid Z_w)$ on Z_w. By definition of $H_{\mathbf{d}}$ in Appendix A.4 and from Equation (24), the map $\Psi_\tau : Z \to H_{\mathbf{d}}$ defined by $\Psi_\tau((\tau, \mathbf{d}), \mathbf{s}) = \mathbf{d} \odot \mathbf{s}$ induces a pmf on $H_{\mathbf{d}}$:

$$\begin{aligned} \hat{P}_{H_{\mathbf{d}}}(\tau)(\mathbf{d} \odot \mathbf{s}) &= P_{Z_w}(\mathbf{s}) \\ w &= (\tau, \mathbf{d}) \end{aligned} \tag{26}$$

The induced pmf implicitly depends[4] on the tournament τ. However we now make the assumptions:

- The scoring probability does not depend on past history.
- The induced pmf $\hat{P}_{H_{\mathbf{d}}}$ is the same as the pmf $P_{H_{\mathbf{d}}}$ discussed in (19)

The probability for T^{k+1} is found from (19),(22), and (23).

Let τ_k denote the current tournament at the k^{th} round, with given pmf written in conventional form as $p^k_{\text{tour}}(\tau_k)$. Then the probability for the tournament at the next round is:

$$\begin{aligned} p^{k+1}_{\text{tour}}(\tau_{k+1}) &= p_{\text{score}}(\mathbf{s} \mid \mathbf{d})\, p_{\text{draw}}(\mathbf{d} \mid \tau_k)\, p^k_{\text{tour}}(\tau_k) \\ p_{\text{score}}(\mathbf{s} \mid \mathbf{d}) &= \prod_{\nu=1}^{\nu=m} p_{\text{grade-score}}(\mathsf{s}_\nu \mid \mathsf{g}_\nu) \\ p_{\text{draw}}(\mathbf{d} \mid \tau_k) &= P_{\mathbb{D}_\tau}(\mathbf{d}) \end{aligned} \tag{27}$$

Hence $p^{k+1}_{\text{tour}}(\tau_{k+1})$ can in principle be constructed for any $k \geq 0$, given initial values at $k = 0$:

$$p^0_{\text{tour}}(\tau_0) = 1, \quad p_{\text{draw}}(\mathbf{d} \mid \tau_0) = 1$$

[4]In real life players' ratings are sometimes affected by earlier results!

B PLAYER SCORE PROBABILITY

The analytic tournament probabilities calculated from Equation (27) enable us to generate the joint distribution for every possible tournament. This can be expressed in the form:

$$
\begin{aligned}
p_{\text{tour}}(\tau_2) &= p(\mathbf{d}_1, \mathbf{s}_1, \mathbf{d}_2, \mathbf{s}_2) \\
\mathbf{d}^r &= (\mathsf{d}_1^r, \cdots \mathsf{d}_m^r) \quad r = 1, 2 \\
\mathbf{s}^r &= (\mathsf{s}_1^r, \cdots \mathsf{s}_m^r) \\
\mathsf{d}_\nu^r &= (i_\nu^r, j_\nu^r) \qquad \nu = 1 \cdots m \\
\mathsf{s}_\nu^r &= (s_{i_\nu}^r, s_{j_\nu}^r)
\end{aligned}
$$

For a given player k and a given tournament τ_2 let $s(k, \tau_2)$ be the sum of the player's scores in the two rounds. Let $T_w^k = \{\tau_2 : s(k, \tau_2) = w\}$. Then the probability that the player scores $w \in \{0, 1, 2\}$ is the sum:

$$p^k(w) = \sum_{\tau_2 \epsilon T_w^k} p_{\text{tour}}(\tau_2)$$

The results of these calculations are given in the Tables 8 and 9.

players	group	grade	mmi	μ	$p_{\text{score}}(0)$	$p_{\text{score}}(1)$	$p_{\text{score}}(2)$
1-4	0	-10	-10	1.0000	0.2500	**0.5**	0.2500
5	1	-7	-7	0.9078	0.2982	0.4958	0.2060
6-8	1	-6	-7	1.0307	0.2339	0.5014	0.2647
9-10	2	-4	-4	0.9141	0.2929	**0.5**	0.2071
11-12	2	-3	-4	1.0859	0.2071	**0.5**	0.2929
13	3	-1	-1	0.6614	0.4431	0.4525	0.1045
14	3	0	-1	0.9160	0.2738	0.5365	0.1897
15-16	3	1	-1	1.2113	0.1416	0.5055	0.3529
17	4	2	2	0.4118	0.6084	0.3713	0.0203
18	4	3	2	0.7793	0.2843	0.6521	0.0636
19	4	4	2	1.1051	0.0939	0.7070	0.1990
20	4	5	2	1.7038	0.0133	0.2696	0.7171

Table 8: Analytic score statistics for 4 player groups

The bold entries in these tables identify the cases where the value of $p^k(1)$ is exactly $\frac{1}{2}$.

players	group	grade	mmi	μ	$p_{\text{score}}(0)$	$p_{\text{score}}(1)$	$p_{\text{score}}(2)$
1-6	0	-30	-30	1.0000	0.2500	**0.5**	0.2500
7	1	-27	-27	0.9543	0.2734	0.4990	0.2277
8-12	1	-26	-27	1.0091	0.2454	0.5001	0.2545
13-14	2	-24	-24	0.9628	0.2689	0.4994	0.2317
15-18	2	-23	-24	1.0186	0.2407	**0.5**	0.2593
19-21	3	-21	-21	0.9712	0.2645	0.4998	0.2357
22-24	3	-20	-21	1.0288	0.2357	0.4998	0.2645
25	4	-18	-18	0.8852	0.3105	0.4937	0.1958
26	4	-17	-18	0.9687	0.2656	0.5002	0.2343
27-30	4	-16	-18	1.0365	0.2316	0.5002	0.2682
31	5	-15	-15	0.8941	0.3055	0.4948	0.1996
32-33	5	-14	-15	0.9767	0.2614	0.5005	0.2381
34-36	5	-13	-15	1.0509	0.2248	0.4995	0.2757
37-38	6	-12	-12	0.9126	0.2950	0.4973	0.2077
39-40	6	-11	-12	0.9985	0.2501	0.5012	0.2486
41-42	6	-10	-12	1.0889	0.2070	0.4971	0.2959
43	7	-9	-9	0.7436	0.3931	0.4702	0.1367
44	7	-8	-9	0.8813	0.3108	0.4972	0.1920
45	7	-7	-9	0.9982	0.2484	0.5049	0.2466
46-48	7	-6	-9	1.1256	0.1885	0.4974	0.3141
49	8	-6	-6	0.7242	0.4050	0.4658	0.1292
50	8	-5	-6	0.8732	0.3144	0.4979	0.1876
51-52	8	-4	-6	1.0175	0.2379	0.5067	0.2554
53-54	8	-3	-6	1.1838	0.1636	0.4890	0.3474
55	9	-3	-3	0.5125	0.5456	0.3963	0.0581
56	9	-2	-3	0.6988	0.4115	0.4781	0.1103
57	9	-1	-3	0.8952	0.2901	0.5246	0.1853
58	9	0	-3	1.1195	0.1783	0.5239	0.2978
59-60	9	1	-3	1.3870	0.0839	0.4452	0.4709
61	10	0	0	0.3578	0.6634	0.3154	0.0212
62	10	1	0	0.5536	0.4986	0.4492	0.0522
63	10	2	0	0.7901	0.3261	0.5576	0.1163
64	10	3	0	1.0751	0.1668	0.5914	0.2418
65	10	4	0	1.4156	0.0528	0.4789	0.4684
66	10	5	0	1.8077	0.0060	0.1802	0.8138

Table 9: Analytic score statistics for 6 player groups